Parallel Bak-Sneppen model and directed percolation

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We introduce a parallel version of the Bak-Sneppen model that is rigorously shown to evolve spontaneously to a site directed percolation critical point. This exemplifies the mechanism underlying a class of self-organized critical models, in which self-organized criticality can be traced back to the existence of an underlying dynamical critical point. This critical point can be reached either by tuning a control parameter or by driving the system infinitely slowly (i.e., by going at a very small rate of evolution). An alternative dynamical evolution formulation of the parallel version of the Bak-Sneppen model enables us to get a very precise estimation of the critical threshold for the corresponding directed percolation problem. [S1063-651X(96)00910-5]

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I. INTRODUCTION

In a recent article [1] we proposed a general conceptual framework to understand self-organized criticality (SOC) as deeply related to tuned out-of-equilibrium critical points, if viewed in an appropriate parameter space. More precisely, SOC was envisionned as resulting from the tuning of the order parameter to a vanishingly small but positive value, thus ensuring that the corresponding control parameter lies exactly at its critical value for the underlying depinning transition. This mechanism was proposed to apply to models of sandpiles, earthquakes, depinning, fractal growth, and forest fires, all examples that have been proposed as exhibiting SOC.

In this paper, we show how this scenario applies quantitatively on a variant of the Bak-Sneppen (BS) model of evolution [2], well known to display SOC (power-law behavior and the existence of a dynamical steady state). The variant uses a modified rule in terms of the parallel evolution of active sites. We are then able to show rigorously that this modified BS model converges precisely onto the critical point of directed percolation (DP). We also introduce a tuned version of the parallel variant of the BS model equivalent to DP, exemplifying the duality between this class of SOC and tunable critical points. Our derivation does not allow us to draw conclusions about the initial version of the BS model, while extensive numerical simulations seem to exclude that it belongs to the DP universality class [3]. We then present a general dynamical evolution equation equivalent to DP, which allows for a precise determination of the critical threshold. The outline of this paper is as follows. Section II describes the original BS model and summarizes the results already known. In Sec. III we introduce a parallel version of the BS model, which we show to be completely equivalent to a site DP problem, though involving an unusual lattice topology. Section IV deals with a very powerful transfer-matrix method for DP, which allows us to obtain very precise estimations of the critical point. It exemplifies the fact that some SOC models spontaneously organize themselves on known tunable dynamical critical points. This extends previously reported cases, such as a dynamical model of rupture in which the rupture pattern self-organizes progressively in the shape of the infinite critical bond percolation cluster and the rupture stops when critical percolation is reached [4].

II. THE EXTREMAL BAK-SNEPPEN MODEL

We study the one-dimensional version of the Bak-Sneppen model of evolution, which is defined as follows. Each of the N sites on a line array is filled with independent random numbers $p_i, i=1,2,\ldots,N$, drawn from a uniform distribution in $[0,1]$. At each step of the dynamics, the smallest p_i in the system is selected and replaced by a new random number p_i' still in $[0,1]$. At the same time, we update its two nearest neighbors p_{i-1}, p_{i+1} with two new, independent random numbers. Periodic boundary conditions are enforced. Note that a choice of a uniform distribution is by no way restrictive since what really matters is the nondecreasing character of the (cumulative) probability distribution function for any probability density. Bak and Sneppen argued that their model is to be thought of as a model of Darwinian biological evolution: each p_i represents the ''fitness'' of the ith species, i.e., its adaptativity to changes and mutations. The species with the lowest p_i dies and its extinction affects its nearest neighbors p_{i-1}, p_{i+1} in the ecological nest, which respond through an instantaneous mutation. Numerical evidence indicates that, in the long-time and large-N limits, the probability density of sites is strictly zero for all $p_i < p_c$ and constant above $p_c = 0.667\,02 \pm 0.000\,08$ [3,5,6]. In the context of statistical mechanics, a more appealing interpretation may be to think of the Bak-Sneppen model as a model for self-organized depinning, where the interface undergoes a

local rearrangement where the force $f_i = 1 - p_i$ is maximal. This was the original motivation of Sneppen [7]: eventually, the interface (which is equal to the set of the f_i's) hardly moves, stuck around the critical point.

SOC models feature not only an asymptotic stationary probability density (the so-called ''steady state''), but also display power-law distributions for ''avalanches.'' In the BS model, these are defined in the following way. Starting from a given $p_{\min}$ at time t, the duration t_0 of an avalanche is the minimum number of time steps t' required for $p_{\min}(t+t')$ to be greater than $p_{\min}(t)$. At the time $t+1$, among the three new random numbers in the system, one or more can be smaller than $p_{\min}(t)$ and the smallest one of all three will initiate subsequent topplings, which follow the same rule at each subsequent time step. This avalanche stops when all sites are above $p_{\min}(t)$. Then an avalanche starts elsewhere, nucleating from the next minimal site, until a steady state is reached where an infinite (the size of an avalanche being defined as the number of toppled sites) avalanche develops: the toppling of the minimal site will eventually trigger an infinite number of topplings among spatially connected sites. Note, however, that the avalanches in this extremal BS model are spatially but not necesarily *spatiotemporally* connected, unlike what happens for DP. This is due to the extremal nature of the dynamics in which an avalanche spreads out only from the minimal site at all time steps and from its two neighbors, while other possible sites being smaller than the $p_{\min}$ that initially triggered the avalanche are left unactivated, until eventually they get modified when the avalanche returns to them.

Two major difficulties have so far precluded an exact analysis. First, there are no evident means of relating the global selection rule and the local cooperative effects. This is bypassed in a mean-field (random neighbor) version where the local spatial connectivity is lost and the problem becomes amenable to an exact analysis through random-walk techniques [8]. The solution predicts $p_c = 1/K$, where K is the number of sites updated at each time step ($K=3$ for the BS model), and the ''usual'' (see, for instance, [9]) $\tau = 3/2$ mean-field SOC exponent for the duration of avalanches, i.e., the probability that an avalanche lasts t time steps scales as $t^{-\tau}$ near p_c. Second, no conservation laws exist to keep track of the local readjustments of the interface. However, if the (random) amount by which the global minimal site is modified is fully reattributed to only one of its neighbor, the model can be exactly solved [10], also by a mapping onto a random-walk problem. Hence it is no wonder that $\tau = 3/2$ is recovered. It is also possible to get exact scaling relations between exponents, writing the time evolution of the ''gap'' $p_{\min}$ and exploiting the hierarchical structure of avalanches [5].

Since an exact solution looks hopeless in this extremal version, we shall review briefly an efficient and reliable method to compute numerically the critical point and the various critical exponents. This method will turn out to be inspiring for the parallel version of the BS model we introduce below. Several authors [3,11] have remarked that it was strictly equivalent to starting from an initial configuration where all but the central site are uniformly distributed between p and 1, for a *fixed* number p in $]0,1[$. The central site, set to any number strictly smaller than p, initiates a p avalanche, which either dies for $p < p_c$ when all sites are greater than p or has a finite probability to go on forever for $p > p_c$. Random numbers used when a site topples are still taken from a uniform density in $[0,1]$. Here p plays the role of a control parameter and each run for a different p corresponds in the original model to starting with a given $p_{\min} = p$. Note that for $p < p_c$, the p avalanches have a nested structure: each p' avalanche is contained within a larger p avalanche for $p' < p$. Reference [3] used this remark to get the following estimates: $\tau = 1.073 \pm 0.003$, $\gamma = 2.70 \pm 0.02$, and $\sigma\nu_\perp = 0.4114 \pm 0.0020$. Here and throughout this paper γ describes the average size of an avalanche near p_c, $\langle s(p) \rangle \sim |p - p_c|^{-\gamma}$, and $\sigma\nu_\perp$ is the exponent describing the maximum spatial (i.e., with respect to the transverse coordinate i) extent of an avalanche: $b(t) \sim t^{\sigma\nu_\perp}$ for an avalanche lasting t time steps, t large, near p_c. The definition of τ is the same as in the mean-field version.

III. CORRESPONDENCE BETWEEN THE PARALLEL BS MODEL AND DIRECTED PERCOLATION

Our parallel BS model is the same as the initial extremal version except that *all* sites that have their numbers below $p_{\min}$, and not solely the smallest one, have their values and those of their neighbors replaced by new random numbers in $[0,1]$. We also introduce the tuned parallel BS model, which naturally consists in updating at a given time step all sites being smaller than p and their (respective) left and right nearest neighbors, where p is fixed in $[0,1]$. The updating is again done in parallel for all unstable sites and any site neighbored by two toppling sites is updated once and only once. p plays the role of a control parameter. Note first that, in contrast to the extremal BS version, avalanches are both spatially and spatiotemporally connected, as in DP.

We first establish the equivalence between the tuned parallel BS model and DP and then show that the parallel BS model self-organizes onto the critical DP point. It is natural to view the time evolution as a two-dimensional (2D) lattice (i,t), $p_i(t)$ being the value of the fitness of the ith site at (discrete)time t. To each site of the 2D lattice, we associate a spinlike variable $n_{i,t}$ equal to 0 if $p_i(t) > p$ and $+1$ if $p_i(t) \leqslant p$. In other words, $n_{i,t} = 0$ if $p_i(t)$ is stable and $+1$ if it is unstable. Obviously, the value of $n_{i,t+1}$ depends on the values of $n_{i-1,t}, n_{i,t}$, and $n_{i+1,t}$: as soon as at least one among these three sites is unstable, the central site $p_i(t)$ will be updated. Since we redraw each site from a uniform distribution between 0 and 1, $p_i(t+1)$ will be smaller than p and thus unstable with probability p and stable with probability $1-p$. This rule determines 14 out of the $2^4 = 16$ *local conditional probabilities* $P(n_{i,t+1} | n_{i-1,t}, n_{i,t}, n_{i+1,t})$, the last two probabilities being $P(0|0,0,0) = 1$ and of course $P(1|0,0,0) = 1 - P(0|0,0,0) = 0$: if a site and its two nearest neighbors are stable, it will remain stable with probability 1. This last condition shows that the tuned parallel BS model is not fully probabilistic: according to the conventional terminology, the phase formed of 0 spins is called an *absorbing* phase [12]. The existence of an absorbing phase is a strong indication that the model should be in the DP universality class [13]. We are going to prove that this is indeed the case. This equivalence was also suggested in [14], but within a continuum limit neglecting spatiotemporal correlations. This

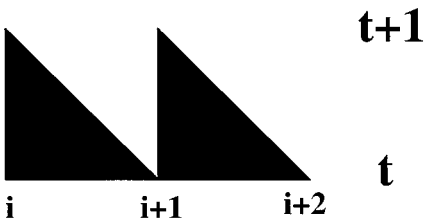

FIG. 1. Two-dimensional lattice on which the plaquettes connecting sites $x_{i,t}, x_{i+1,t}$ to site $x_{i,t+1}$ are represented. The topology is that of the square lattice, for which $p_c = 0.705\ldots$.

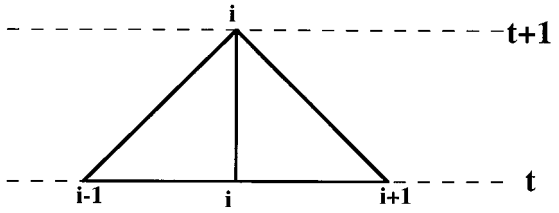

FIG. 2. Plaquettes of our three-partner parallel BS model, showing the rather non-standard structure of the lattice topology, which is important to recognize for a correct correspondence with a DP problem.

rather uncontrolled approximation led to the erroneous assertion that the arguments would hold for the original extremal model.

For the sake of pedagogy, we first consider a two-sided model where just the left site itself and its neighbor of an unstable site are updated at the next time step. The main advantage is that we can compare the value of the singularity that develops in the probability distribution with the well-documented value of the directed percolation threshold for a 2D *square* lattice: $p_c = 0.705\,489 \pm 0.000\,004$ [15]. The local conditional probabilities read $P(0|1,0) = P(0|0,1) = P(0|1,1) = 1-p$ and $P(0|0,0) = 1$ [supplemented, of course, with $P(0|n_1,n_2) = 1 - P(1|n_1,n_2)$ for any n_1, n_2]. These local conditional probabilities are defined on every other triangular ''plaquettes,'' say, the up-pointing ones if time is running upward as in Fig. 1, which connects causal spins from two successive time slices. Now imagine that we call 0 spins *dry* sites and 1 spins *wet* sites; the rules of the game are that sites have probability p_s to be wet and that up-pointing bonds (the diagonal edges of a triangle) have probability p_b to conduct fluid. This is just a generalized mixed bond-site percolation problem (a particular case of the Domany-Kinzel automaton), where the local conditional probabilities are straightforward to write down [12,16]:

$$P(1|0,0) = 0,$$

$$P(1|1,0) = P(1|0,1) = p_s p_b,$$

$$P(1|1,1) = p_s[{p_b}^2 + 2p_b(1-p_b)] = p_s p_b(2-p_b).$$

We can now readily proceed to the identification: one has $p = p_s p_b$ and $p = p_s p_b(2-p_b)$, so that $p = p_s$ and $p_b = 1$. The one-sided tuned parallel BS is thus completely isomorphic to directed site percolation on a square lattice. Such an equivalence generalizes readily for the original (three sites) tuned parallel BS model [17], but the lattice topology of the corresponding site DP problem is rather nonstandard (see Fig. 2). This subtlety has been overlooked in the literature, where some authors have compared their values of the three-partner BS critical threshold with that of a *bond* DP on a *square* lattice: $0.6447\ldots$. Let us mention that the DP threshold corresponding to the lattice topology shown in Fig. 2 and to the original (three-partner) tuned parallel BS model can be easily obtained numerically through the transfer-matrix method discussed in Sec. IV. It yields $p_c' = 0.5371 \pm 0.0001$ further evidence that the original BS model is *not* a DP problem.

From the definition of the tuned parallel BS model, it is clear that the control parameter p can be imposed to be larger than the DP threshold, corresponding to the depinned regime, where the fraction v of unstable or active sites (i.e., those with $p_i < p$) becomes nonzero. v is the *order parameter* for this depinning transition: $v \propto (p - p_c')^{\beta}$, where $\beta = 0.2764 \pm 0.0008$.

As already asserted, the DP critical state can be reached either by tuning a control parameter as in the tuned parallel BS model or by going to a very small rate of evolution, since all that really matters is how avalanches are defined. We now show that the parallel BS model converges indeed to the DP critical point. Let us denote $p_{\min}(t_0)$ the gap, i.e., the smallest number in the lattice at time t_0. Suppose that $p_{\min}(t)$ converges at large times to a value p_∞ larger that the DP critical value p_c. Consider a time t_0 at which an avalanche starts with $p_c < p_{\min}(t_0) < p_\infty$. Since the avalanche develops according to the parallel updating rule, nothing distinguishes its time evolution from the dynamics of the tuned parallel BS model with p fixed to $p_{\min}(t_0)$. However, since the tuned parallel BS model is strictly equivalent to DP, the condition $p_c < p_{\min}(t_0) = p$ implies that the avalanche is infinite. As a consequence, $p_{\min}(t_0)$ cannot be less than p_∞, in contradiction with the hypothesis. Suppose conversely that $p_\infty < p_c$ of DP. Let us consider an avalanche starting at $p_{\min}(t) = p_\infty$. Again, since the avalanche develops according to the parallel updating rule, nothing distinguishes its time evolution from the dynamics of the tuned parallel BS model with p fixed to p_∞. But since p_∞ is supposed smaller than the DP p_c, the activity must die after a finite number of time steps, in contradiction with the starting hypothesis $p_{\min}(t) = p_\infty$. We are thus led to conclude that $p_\infty = p_c$ of DP. This correspondence allows us to obtain the exponent of the avalanche distribution from DP through standard scaling arguments [3,11]: $\tau = 3 - 1/(1+\beta/\gamma) = 2.1082 \pm 0.0004$, where τ is defined by $P(S)dS \sim S^{-\tau}dS$, where S is the size of an avalanche defined here as the size of the corresponding DP cluster. $\gamma = 2.278 \pm 0.002$ and $\sigma\nu_\perp = \nu_\perp/(\gamma+\beta) = 0.4294 \pm 0.0006$. γ is the standard susceptibility exponent for percolation problems, while $\nu_\parallel = 1.7339 \pm 0.0003$ and $\nu_\perp = 1.0969 \pm 0.0003$ describe the divergence of the two correlation lengths. All the values quoted here for the DP exponents are

taken from Ref. [15]. Grassberger has proposed a different modification of the BS model, also in the DP universality class, in which a single site is activated (not that with minimal fitness), but randomly among all active sites [3].

Note that the gap equation, derived for the usual extremal BS model [5,18], also holds for our parallel BS model. The gap equation defines the mechanism of approach to the self-organized critical attractor, whatever it might be. In contrast, the Γ equation does not hold for the parallel BS model since it is based on the hierarchical structure of embedded avalanches, holding true due to the extremal rule of the standard BS model.

IV. SELF-ORGANIZED FORMULATION OF DIRECTED PERCOLATION

We have thus seen that two modified versions of the BS model belong to the DP universality class. Can other self-organized models be invented that exhibit similar properties? From a different perspective, note that the tuned parallel BS model describes a dynamical evolution equivalent to the probabilistic formulation of the DP model. This is quite reminiscent of the correspondence between the Fokker-Planck formalism (probabilistic equation as for DP) and the Langevin formalism (explicit stochastic evolution equation as for the tuned parallel BS model). Can we imagine other stochastic evolution equations that are, in some sense, equivalent to DP? The philosophy of our approach is to transform a probabilistic *discrete* process into a deterministic *continuous* one, via the introduction of a random function [19]. By discrete and continuous we refer to the ensembles in which the (fitness) variables $x_{i,t}$ attributed to each site i,t take their values. In both cases, these variables live on a discrete space-time lattice. To be equivalent to the site DP on the square lattice, we need an equation of evolution for the $x_{i,t}$ that can be easily connected to the DP connectivity probability p (fixed in $]0,1[$). To do this, we assume that a site i,t is wet (dry) if its variable $x_{i,t}$ is less (larger) than p. Note that this rule has already been used for the tuned parallel BS model and is again an essential step to achieve the desired correspondence. Consider the two sites i,t and $i-1,t$ that are connected to site $i,t+1$ in the square lattice. The sought evolution equation is a formula specifying $x_{i,t+1}$ as a function of $x_{i,t}$ and $x_{i-1,t}$. The rules are the following: if $x_{i,t}$ and $x_{i-1,t}$ are both larger than p (dry), then the site $i,t+1$ must be empty according to the DP rule, i.e., $x_{i,t+1}$ must be larger than p; if $x_{i,t}$ or $x_{i-1,t}$ or both are less than p (wet), then the site $i,t+1$ is wet with probability p, i.e., $x_{i,t+1}$ is smaller than p with probability p.

These rules are embodied by the evolution equation

$$x_{i,t+1}=\max[\,\eta_{i,t+1},\min(x_{i,t},x_{i-1,t})], \tag{1}$$

where the η's are random numbers, independently drawn for each site i,t from a uniform density between 0 and 1. Equation (1) provides the sought stochastic equation of evolution, equivalent to the DP probabilistic model.

We have gained something in the construction process: indeed, notice that p does not appear explicitly in the evolution equation (1). It is thus equivalent to DP for an *arbitrary* p. In other words, following the evolution equation (1) with arbitrary initial conditions allows one to simulate DP for all p's at the same time. The same set of $x_{i,t}$ allows one to reconstruct the cluster statistics for all p's: for $p<p_c$, only isolated finite clusters of sites with $x_{i,t}<p$ exist (nonpercolating phase), while for $p>p_c$, an infinite cluster of sites with $x_{i,t}<p$ appears (percolating phase). This situation is remarkable in the sense that there is no control parameter for the time evolution of the $x_{i,t}$. The parameter p serves only as a threshold to distinguish wet from dry sites. This is completely analogous to the ''sea-level'' formulation of (nondirected) percolation: a random number between 0 and 1 is attributed to each lattice site. One then introduces an arbitrary p (the sea level) and selects all sites whose number is less than p. For $p<p_c$ (where p_c is the percolation threshold), only isolated finite lakes exist, whereas for $p>p_c$, an infinite ocean appears.

There is, however, an important difference between standard percolation and directed percolation: in standard percolation, the $x_{i,t}$ are uniformly distributed and do not provide any information on the critical properties; in directed percolation corresponding to Eq. (1), the $x_{i,t}$ have a distribution $P(x)$ that is singular at the DP threshold $x=p_c$ (in the long-time limit). The evolution equation (1) thus describes the subtle long-range correlation that is intrinsic of DP. It turns out that (1) is implicitly used by many authors, since it basically amounts to implementing an exclusive or. It has a great advantage over alternative schemes in that a single run simulates many values of p at the same time. It is thus worthwhile to explain in depth how the DP critical behavior can emerge from the evolution of the distribution of the x_i's. To see this, simply note that

$$\rho(p)=\int_0^p P(x)dx, \tag{2}$$

where $\rho(p)$ is the density of DP growth sites. From the known singular properties of $\rho(p)$ [20], we deduce that near p_c we should have the scaling behavior

$$P(x)\approx t^{1/\nu_\parallel-\delta}g((x-p_c)t^{1/\nu_\parallel}),$$

where $\nu_\parallel=1.7336$ and $\delta=0.1596$, leading to $P(x)\approx(x-p_c)^{-(1-\delta\nu_\parallel)}\approx(x-p_c)^{-0.7233}$ for large t. This stems from the relation $\rho(p)\approx(p-p_c)^\beta F((p-p_c)t^{1/\nu_\parallel})\approx(p-p_c)^\beta$ for t large and using the scaling relation $\beta=\delta\nu_\parallel$ [20]. Equation (1) provides an extremely efficient method for a fast numerical estimation of the percolation threshold for site DP on a square lattice: we find $p_c=0.7056\pm0.0002$, in agreement with the value obtained by much more complicated series expansions in [15]. Used in combination with the leaf-tree algorithm [20], one can also get the critical exponents. Furthermore, all the results obtained above are not specific of the $(1+1)$-dimensional case and generalize straigthforwardly to higher space dimensions as well as to directed bond percolation for any spatial connectedness (see [21] for an application to $2+1$ DP).

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